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# ON THE UNLOADING PROCESS FOR CONTACT INTERACTION*

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The unloading process in a body under the action of a stamp is investigated. It is assumed that the unloading occurs at all points of the body. The contact area between the body and the stamp can change during the unloading; consequently, the unloading problem during contact interaction is non-linear. A generalization to the case of contact problems is proposed for the theorem of unloading /1/. A variational principle is obtained in the unloading displacements, and the existence and uniqueness of the solution of the unloading problem are investigated. The unloading process is examined in an elastic-plastic half-space on which a stamp of circular planform acts. The change in the contact area and in the contact stresses during unloading is studied, and the shape of the residual impression is obtained. The problem is investigated by using the Galin solution /2/ of the action of a circular stamp and a load applied outside the stamp on an elastic half-space. Numerical methods of solving contact problems with unloading are also examined; an example is presented for the numerical solution of the problem of plane deformation in the compression of a strip by two stamps with subsequent unloading.

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**1. Formulation of the problem.** The unloading process is investigated in a body occupying the domain $\Omega$ of three-dimensional space bounded by a piecewise-smooth surface $\Gamma$. The deformations and displacements are considered to be small. We associate a monotonically increasing parameter $t$, $t \in [0, T]$, which we call the time, with the deformation process. We understand $u_i(x, t)$, $\varepsilon_{ij}(x, t)$, and $\sigma_{ij}(x, t)$ to be components of the displacement vector, the strain and stress tensors at the point $x = (x_1, x_2, x_3)$ at the time $t$. The surface $\Gamma$ can consist of three parts: $\Gamma_u$, $\Gamma_\sigma$, $\Gamma_c$. The displacements $U_i(x, t)$ are given on the part $\Gamma_u$, and the forces $S_i(x, t)$ on the part $\Gamma_\sigma$. A rigid stamp, whose shape and location at the time $t$ will be described by the distance $\Phi(x)$ from the point $x$ to the stamp surface measured along the normal direction $\nu(x)$, acts on the body at points of the surface $\Gamma_c$. The contact area is not given in advance and can change during the deformation. It is assumed that there is no friction onthe contact area. We let $u_\nu$, $u_\tau$, $\sigma_\nu$, $\sigma_\tau$ denote the normal and tangential components of the displacement and stress vectors on $\Gamma_c$. Then stamp interaction with the body $\Omega$ is described by the following conditions /3,4/:

$$\begin{gathered}\sigma_\nu(x, t) \leqslant 0, \ \sigma_\tau(x, t) = 0 \\ u_\nu(x, t) \leqslant \Phi(x, t), \ \Phi(x, 0) \geqslant 0 \\ \sigma_\nu(x, t)[u_\nu(x, t) - \Phi(x, t)] = 0, \ \forall x \in \Gamma_c, \ \forall t \in [0, T]\end{gathered} \tag{1.1}$$

Let the functions $U_i(x, t)$, $S_i(x, t)$, $\Phi(x, t)$ be such that starting with the time $t^*$ unloading occurs at all points of the body $\Omega$. It is assumed that for $t \leqslant t^*$ the solution of the problem is obtained within the framework of a certain definite theory of plasticity, while at the time $t^*$ the condition of beginning of the unloading associated with this theory is used. We introduce increments of the unloading displacements, deformations, and stresses $\Delta u_i$, $\Delta\varepsilon_{ij}$, $\Delta\sigma_{ij}$ in conformity with the relationships

$$\begin{gathered}u_i(x, t) = u_i(x, t^*) + \Delta u_i(x, t) \\ \varepsilon_{ij}(x, t) = \varepsilon_{ij}(x, t^*) + \Delta\varepsilon_{ij}(x, t) \\ \sigma_{ij}(x, t) = \sigma_{ij}(x, t^*) + \Delta\sigma_{ij}(x, t)\end{gathered} \tag{1.2}$$

The increments $\Delta U_i(x, t)$, $\Delta S_i(x, t)$, $\Delta\Phi(x, t)$ are defined analogously. We consider the increments of the stress and strain tensor components during unloading to be connected by the linear relationships

$$\Delta\sigma_{ij} = A^*_{ijkl}(x)\, \Delta\varepsilon_{kl} \tag{1.3}$$

The elastic constants $A^*_{ijkl}(x)$ generally depend on the history of the deformation before the time $t^*$.

Therefore, the problem of determining the state of stress and strain during unloading involves constructing the functions $u_i$, $\varepsilon_{ij}$, $\sigma_{ij}$ that satisfy the equilibrium equations, the Cauchy relations and the relationships (1.2) and (1.3), as well as the boundary conditions on $\Gamma_u$ and $\Gamma_\sigma$ and conditions (1.1) on the contact surface.

**2. Theorem on unloading during contact interaction.** The solution of boundary value problems on unloading for $\Gamma_c = \varnothing$ is based on Il'yushin's theorem /1/, according to which the state of stress and strain is determined by the relations (1.2) at an arbitrary unloading time, while the increments $\Delta u_i$, $\Delta\varepsilon_{ij}$, $\Delta\sigma_{ij}$ are a solution of a boundary value problem of elasticity theory for the domain $\Omega$ for displacements $\Delta U_i(x, t)$ given on $\Gamma_u$ and forces $\Delta S_i(x, t)$ given on $\Gamma_\sigma$. The formal replacement of the quantities in conditions (1.1) by their increments can obviously result in violation of these conditions; consequently, the theorem on unloading cannot be carried over directly to the contact interaction problem. Such a deduction is explained by the non-linear nature of the contact problems with indefinite contact areas even if linear relationships are used to connect the stress and strain.

Let us generalize the theorem on unloading to contact problems in such a way as to conserve relationships (1.2). To this end, we formulate the following special conditions for the increments on the surface $\Gamma_c$:

$$\begin{gathered}\Delta\sigma_\nu(x, t) \leqslant -\sigma_\nu(x, t^*), \ \Delta\sigma_\tau(x, t) = 0 \\ \Delta u_\nu(x, t) \leqslant -u_\nu(x, t^*) + \Phi(x, t) \\ [\Delta\sigma_\nu(x, t) + \sigma_\nu(x, t^*)][\Delta u_\nu(x, t) + u_\nu(x, t^*) - \Phi(x, t)] = 0, \ \forall x \in \Gamma_c, \ \forall t \in [t^*, T]\end{gathered} \tag{2.1}$$

It can be seen that if $\Delta u_i$, $\Delta\sigma_{ij}$ satisfy conditions (2.1), then $u_i$, $\sigma_{ij}$ defined by (1.2), will satisfy conditions (1.1). Therefore, we obtain the following theorem on unloading during contact interaction.

*Theorem 1.* To determine the state of stress and strain during unloading for contact interaction between a body and a stamp, it is sufficient to solve the elasticity theory problem for the domain $\Omega$ with respect to $\Delta u_i$, $\Delta\varepsilon_{ij}$, $\Delta\sigma_{ij}$ by replacing $U_i$, $S_i$ in the boundary conditions by $\Delta U_i$, $\Delta S_i$ and taking the conditions (2.1) on the contact surface. The displacement vector and strain and stress tensor components are determined by relationships (1.2).

*Corollary.* If a stamp acting on the boundary of an elastic-plastic half-space is moved translationally in the normal direction to the boundary, and the contact stresses at the time of the beginning of the unloading are limited, then the beginning of the unloading is accompanied by retardation of the surfaces making contact at points of the contact-area contour.

Let the contact area be reduced during an arbitrarily small time interval $\Delta t$ following $t^*$. Then it follows from Theorem 1 that the increments of the contact stresses $\Delta\sigma_\nu$ will equal the contact stresses, with opposite sign, during impression of a stamp with a flat base (the contact area does not change) or with a concave base (the contact area increases during the time $\Delta t$) to a depth $\Delta\Phi$ in an elastic half-space. In both cases the contact stresses on the stamp edges will be unbounded for arbitrarily small $\Delta\Phi$, which results in violation of the condition $\sigma_\nu(x, t^*) + \Delta\sigma_\nu(x, t) \leqslant 0$ because of the boundedness of $\sigma_\nu(x, t^*)$.

We note that for a sufficiently high degree of initial loading, the residual stresses that occur can cause secondary plastic deformations, for instance, when a sphere is pressed into a half-space /5/. In such cases Theorem 1 is applicable only up to the occurrence of the secondary plastic deformations. Since the corollary of Theorem 1 refers to the time of the beginning of unloading, the circumstance noted does not restrict the applicability of this corollary.

## 3. Variational formulation of the unloading problem.

We use the Sobolev space $H \equiv [W_2^1(\Omega)]^3$ of the vector functions $v(x) = (v_1(x), v_2(x), v_3(x))$ defined in the domain $\Omega$ and square-summable in $\Omega$ together with their first partial derivatives. We understand the scalar product in $H$ to be

$$(u, v)_H = \int_\Omega (u_i v_i + u_{i,j} v_{i,j})\, d\Omega$$

We introduce the set $V^*(t) \subset H$ of kinematically possible increments of the displacements $\Delta v \in H$ in which we include increments satisfying the boundary conditions on $\Gamma_u$ and the kinematic conditions from (2.1) on $\Gamma_c$

$$V^*(t) = \{\Delta v \in H \mid \Delta v_i(x, t) = \Delta U_i(x, t),\ \forall x \in \Gamma_u$$
$$\Delta v_\nu(x, t) \leqslant -u_\nu(x, t^*) - \Phi(x, t),\ \forall x \in \Gamma_c\}$$

We denote by $\Delta\zeta_{ij}$ the strain increments corresponding to $\Delta v$ according to the Cauchy relationships. As in /4/, we obtain the following integral inequality by using Gauss's theorem:

$$\int_\Omega \Delta\sigma_{ij}(\Delta\zeta_{ij} - \Delta\varepsilon_{ij})\, d\Omega - \int_{\Gamma_\sigma} \Delta S_i(\Delta v_i - \Delta u_i)\, d\Gamma - \int_{\Gamma_c} \Delta\sigma_\nu(\Delta v_\nu - \Delta u_\nu)\, d\Gamma = 0, \quad \forall \Delta v \in V^*(t), \quad \forall t \in [t^*, T]$$

Let $\sigma_\nu^* = \sigma_\nu(x, t^*)$, $u_\nu^* = u_\nu(x, t^*)$, and we transform the integrand in the last term into

$$\Delta\sigma_\nu(\Delta v_\nu - \Delta u_\nu) = \sigma_\nu(v_\nu - u_\nu^* - u_\nu + u_\nu^*) - \sigma_\nu^*(\Delta v_\nu - \Delta u_\nu) = \sigma_\nu(v_\nu - u_\nu) - \sigma_\nu^*(\Delta v_\nu - \Delta u_\nu)$$

It follows from (1.1) that $\sigma_\nu(v_\nu - u_\nu) \leqslant 0$, $\forall v_\nu = u_\nu^* + \Delta v_\nu$, $\Delta v_\nu \in V^*(t)$. Then by using relationship (1.3), we obtain the following variational inequality:

$$\int_\Omega A^*_{ijkl}\Delta\varepsilon_{kl}(\Delta\zeta_{ij} - \Delta\varepsilon_{ij})\, d\Omega - \int_{\Gamma_\sigma} \Delta S_i(\Delta v_i - \Delta u_i)\, d\Gamma + \tag{3.1}$$
$$\int_{\Gamma_c} \sigma_\nu^*(\Delta v_\nu - \Delta u_\nu)\, d\Gamma \geqslant 0, \quad \forall \Delta v \in V^*(t), \quad \forall t \in [t^*, T]$$

If the quadratic form $A^*_{ijkl}\Delta\varepsilon_{ij}\Delta\varepsilon_{kl}$ is positive, then the variational inequality (3.1) is equivalent to the following extremal problem /6/:

$$\inf_{\Delta v \in V^*(t)} \left\{J(\Delta v) = \frac{1}{2} B(\Delta v, \Delta v) - F(\Delta v) - F^*(\Delta v)\right\} \tag{3.2}$$
$$B(\Delta v, \Delta v) = \int_\Omega A^*_{ijkl}\Delta\zeta_{ij}\Delta\zeta_{kl}\, d\Omega$$
$$F(\Delta v) = \int_{\Gamma_\sigma} \Delta S_i \Delta v_i\, d\Gamma, \quad F^*(\Delta v) = \int_{\Gamma_c} \sigma_\nu^* \Delta v_\nu\, d\Gamma$$

By using well-known methods /6/, it can be shown that the solution of the variational inequalities (3.1) (or the extremal problem (3.2)) is a generalized solution of the problem in the original formulation.

## 4. Existence and uniqueness of the solution.

Starting from the variational formulation of the problem, we now turn our attention to the fact that the functional $J(\Delta v)$ is quadratic and can be considered as a functional of the total energy for a certain linear elasticity theory problem with addditional forces $-\sigma_\nu^*$ given on $\Gamma_c$. Moreover, the set $V^*(t)$ is convex and closed for all $t \in [t^*, T]$. In this case, it is sufficient to use appropriate general theorems /7/ in investigating the existence and uniqueness of the solution. As in /4/, we introduce the auxiliary set $V_0^*(t)$ which is obtained by shifting all the elements of the set $V^*(t)$ by a fixed element $u_0 \in V^*(t)$ satisfying the conditions

$$\Delta u_{0i}(x, t) = \Delta U_i(x, t), \ \forall x \in \Gamma_u$$
$$\Delta u_{0\nu}(x, t) = -u_\nu(x, t^*) + \Phi(x, t), \ \forall x \in \Gamma_c$$

We also introduce the subspace $R \subset H$ of displacements of the body $\Omega$ as a rigid body. Then the following assertion holds as a special case of the theorems of the existence and uniqueness of the solution for linear unilateral problems /7/:

*Theorem 2.* Let the functions $\sigma_\nu^*(x)$, $A_{ijkl}^*(x)$ satisfy the conditions

$$\sigma_\nu^* \in H^{-1/2}(\Gamma_c), \ A_{ijkl}^* \in L^\infty(\Omega)$$
$$A_{ijkl}^* \Delta\varepsilon_{ij}\Delta\varepsilon_{kl} \leqslant \alpha\Delta\varepsilon_{ij}\Delta\varepsilon_{ij}, \ \alpha > 0$$

and the given functions $U_i$, $S_i$, $\Phi$ by subject to the requirements

$$U_i \in H^{1/2}(\Gamma_u), \quad S_i \in H^{-1/2}(\Gamma_\sigma), \quad \Phi \in H^{1/2}(\Gamma_c), \quad \forall t \in [t^*, T]$$

and for all $\Delta r \in R \cap V_0^*(t)$ let the inequality

$$F(\Delta r) - F^*(\Delta r) \leqslant 0, \ \forall t \in [t^*, T] \tag{4.1}$$

hold, where the equality sign holds only for such $\Delta r \in R \cap V_0^*(t)$ for which $-\Delta r \in R \cap V_0^*(t)$. Then a solution $\Delta u \in H$, $\forall t \in [t^*, T]$ exists of the problem about the unloading process during contact interaction between a body and a stamp that is unique to within increments of the displacement $\Delta r \in R$ such that $F(\Delta r) - F^*(\Delta r) = 0$.

Note that conditions (4.1) are necessary only for $\Gamma_u = \varnothing$.

**5. On the interaction between a circular stamp and a half-space in the unloading process.** In an elastic-plastic half-space $x_3 \leqslant 0$ let a stamp of circular planform with flat base of radius $c$ be embedded to a depth $\Phi^*$, and then let the depth of embedding decrease monotonically. It is assumed there is no friction between the body and stamp surfaces. The contact pressure distribution $p^*(x_1, x_2)$ at the time $t^*$ is considered to be known. Assuming the elastic-plastic deformation prior to $t = t^*$ did not change the elastic constants, the elastic modulus $E$, and Poisson's ratio $\nu$, we determine the size of the contact area and the contact stress distribution as a function of the stamp position during unloading. We also find the profile of the residual impression.

The problem of impressing a stamp of circular planform into a rigidly plastic half-space was investigated in /8,9/ using the total plasticity condition. The numerical solution of the corresponding problem for an elastically ideal plastic medium is proposed in /10/.

We introduce the cylindrical coordinate system $(r, \varphi, z)$ by locating the origin at the centre of the circle of initial contact. The direction of the $0z$ axis is in agreement with the direction of $0x_3$ axis. Let the depth of stamp embedding diminish by $\Delta\Phi$ as compared with $\Phi^*$ up to a certain time of unloading. According to Theorem 1, the pressure distribution $p(r)$ can be represented for the depth of embedding $\Phi^* - \Delta\Phi$ in the form

$$p(r) = p^*(r) - \Delta p(r), \ \ r \leqslant c$$

where $\Delta p(r) = -\Delta\sigma_{zz}(r, \varphi, 0)$ is the normal stress on the surface $z = 0$ corresponding to the solution of the elasticity-theory problem for a half-space under the following boundary conditions:

$$-\Delta\sigma_{zz}(r, \varphi, 0) = \Delta p(r) \leqslant p^*(r) \tag{5.1}$$
$$\Delta\sigma_{rz}(r, \varphi, 0) = \Delta\sigma_{z\varphi}(r, \varphi, 0) = 0, \ -\Delta u_z(r, \varphi, 0) \geqslant \Delta\Phi$$
$$[\Delta p(r) - p^*(r)][\Delta u_z(r, \varphi, 0) + \Delta\Phi] = 0, \ \forall r \leqslant c$$
$$\Delta\sigma_{zz}(r, \varphi, 0) = \Delta\sigma_{rz}(r, \varphi, 0) = \Delta\sigma_{z\varphi}(r, \varphi, 0) = 0, \ \forall r > c \tag{5.2}$$

Considering that the contact stresses at the beginning and during unloading are limited, we apply the corollary of Theorem 1 to each time of unloading and obtain that the contact area during unloading is a circle with monotonically decreasing radius $a \leqslant c$. Then conditions (5.1) can be replaced by conditions in the form of the equalities

$$\Delta p(r) = p^*(r), \ a \leqslant r \leqslant c \tag{5.3}$$

$$-\Delta u_z(r,\varphi,0) = \Delta\Phi,\ r \leqslant a$$

$$\Delta\sigma_{rz}(r,\varphi,0) = \Delta\sigma_{z\varphi}(r,\varphi,0) = 0,\ r \leqslant c$$

Considitions (5.2) and (5.3) correspond to the elasticity-theory problem about the action of a circular stamp with a flat base of radius $a$ and axisymmetric load distributed around the ring $a \leqslant r \leqslant c$ on the boundary of a half-space. Such a problem is a special case of the Galin problem /2/ concerning the action of a stamp of circular planform and a normal load distributed outside the stamp. Using Galin's solution, we conclude that the quantity $\Delta p(r)$ can be represented in the form

$$\Delta p(r) = \Delta p_1(r) + \Delta p_2(r) \tag{5.4}$$

where $\Delta p_1(r)$ is the pressure under a circular stamp with base of radius $a$ upon impression to a depth $\Delta\Phi$ equal to

$$\Delta p_1(r) = \frac{E}{\pi(1-\nu^2)} \cdot \frac{1}{\sqrt{a^2-r^2}}\Delta\Phi \tag{5.5}$$

and $\Delta p_2(r)$ is the additional pressure that occurs under a stamp of radius $a$ due to the action of the load $p^*(r)$ distributed over the ring $a \leqslant r \leqslant c$ and equal to

$$\Delta p_2(r) = -\frac{1}{\pi^2}\int_a^c\int_0^{2\pi}\frac{p^*(\rho)}{\rho^2+r^2-2\rho r\cos\vartheta}\sqrt{\frac{\rho^2-a^2}{a^2-r^2}}\,\rho\,d\rho\,d\vartheta \tag{5.6}$$

The radius of the contact area $a$ is determined from the condition of continuity of the pressure on the contour of the contact area: $p(a) = 0$. Determination of the residual impression reduces to determining the displacements of points of the circle $r \leqslant c$ that occur due to the action of the normal load $p^*(r)$ distributed over this circle. Using the Boussinesq solution, we obtain the profile of the residual impression

$$w(r) = -\Phi^* + \frac{1-\nu}{\pi E}\int_a^c\int_0^{2\pi}\frac{p^*(\rho)}{\sqrt{\rho^2-r^2-2\rho r\cos\vartheta}}\,\rho\,d\rho\,d\vartheta \tag{5.7}$$

Therefore, the contact stress distribution during unloading and the shape of the residual impression for any pressure distribution $p^*(r)$ at the time of the beginning of unloading has been obtained in quadratures. The approach elucidated can also be utilized in the case of the action of stamps of circular planform with non-planar base.

In the case $p^*(r) = p^* = \text{const}$, simple expressions are successfully obtained for the radius of the contact area, the contact stresses, and the profile of the residual impression. We note that the pressure distribution $p^*(r)$ differs by not more than 17% /8,9/ from the constant value according to the scheme of a rigidly plastic body.

For $p^*(r) = \text{const}$ we obtain from (5.6)

$$\Delta p_2(r) = \frac{2p^*}{\pi}\operatorname{arc\,tg}\sqrt{\frac{c^2-a^2}{a^2-r^2}} - \frac{2p^*}{\pi}\sqrt{\frac{c^2-a^2}{a^2-r^2}}$$

The condition of continuity of the pressure on the contour of the contact area $p^* - \Delta p(a) = 0$ will be satisfied if the radius of the contact area is selected as follows:

$$a = \left\{c^2 - \left[\frac{E}{2p^*(1-\nu^2)}\Delta\Phi\right]^2\right\}^{1/2} \tag{5.8}$$

Formula (5.8) is meaningful for $\Delta\Phi \leqslant 2p^*c(1-\nu^2)/E$; for $\Delta\Phi = 2p^*c(1-\nu^2)/E$ total separation of the stamp from the half-space occurs. By taking the radius of the contact area as given by (5.8), we obtain the contact pressure distribution at an arbitrary time of the unloading process

$$p(r) = \begin{cases} p^* - \dfrac{2p^*}{\pi}\operatorname{arc\,tg}\sqrt{\dfrac{c^2-a^2}{a^2-r^2}}, & r \leqslant a \\ 0, & r \geqslant a \end{cases}$$

The profile of the residual impression is described for $p^*(r) = \text{const}$ by the function

$$w(r) = -\Phi^* + \frac{4(1-\nu^2)}{\pi E}p^*c\mathbf{E}\left(\frac{r}{c}\right)$$

where $\mathbf{E}(\ldots)$ is the complete elliptic integral of the second kind.

**6. Example of the numerical solution.** Using the variational formulation (Sec.4), a method was developed for the numerical solution of unloading problems under plain strain conditions. The extremal problem (3.3) discretized using the method of finite elements, while the solution of the non-linear programming problem that occurs was obtained by the generalized method of sequential upper relaxation /11/. A set of programs was developed for investigating the state of stress and strain during the unloading process for plain strain of a multilayered packet.

As an example, we consider the problem of the compression of a strip of rectangular shape $-2h \leqslant x_1 \leqslant 2h$, $-h \leqslant x_2 \leqslant h$ in a section of the plane $0x_1x_2$ by two stamps. The stamp surfaces are described by the following equations (because of symmetry the equations are presented only for the upper stamp):

$$\frac{x_2}{h} = 1 + \frac{\tau_s}{2G}\left(\frac{x_1}{h}\right)^2 \tag{6.1}$$

$$\frac{x_2}{h} = 1 + \frac{\tau_s}{8G}\left(\frac{x_1}{h}\right)^2 \tag{6.2}$$

$$\frac{x_2}{h} = 1 + \frac{2\tau_s}{G}\left[1 - \frac{4}{(x_1/h)^2 + 4}\right] \tag{6.3}$$

where $G$ is the elastic modulus for torsion, and $\tau_s$ is the elastic limit for torsion. As a result of the monotonic growth of the load the stamps come together to the distance $2h - 2\Phi^*$, $\Phi^* = 1.5\,\tau_s h/G$, and then the stamp is released from the load, which results in unloading in the strip. It is assumed that the stamps shift transversally in the direction of the $0x_2$ axis under active loading and unloading.

The theory of small elastic-plastic strains was utilized in investigating the active loading process; a linear hardening scheme with ratio 0.05 between the elastic and tangential moduli was used. Poisson's ratio was 0.3. The problem was solved numerically under active loading by using the method described in /12/.

The unloading problem was solved under the assumption that unloading occurs at all points of the strip and secondary plastic strains do not occur. This assumption was confirmed by the solution of the problem.

The contact stress distribution during unloading is represented in Fig. 1 for the cases of compression of stamps with the Eqs.(6.1) for the surfaces (continuous curves) or (6.3) (dashed curves). Curves *1–4* correspond to the spacing between the stamps $2h - 2\Phi$, $\Phi G/\tau_s h = 1.5$; 1.3, 1.1, 0,9. We note that the contact stress diagrams are similar in a sufficiently large range of variation of $x_1$, where the maximum value of the contact pressure is a linear function of the closure $2\Delta\Phi$.

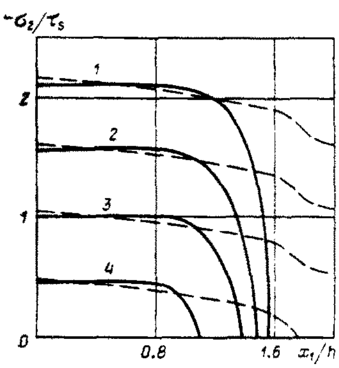


Fig.1

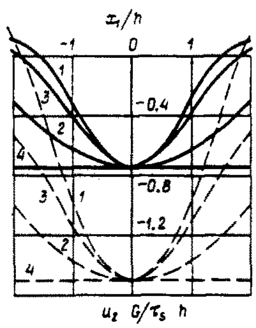


Fig.2

Vertical displacements $u_2(x_1)$ of the surface $x_2 = h$ are presented in Fig. 2 at the time of the beginning of unloading (dashed curves) and in the residual state (continuous curves). The numbers *1, 2, 3* correspond to stamps with the equations (6.1), (6.2), (6.3) for the surfaces. For comparison, Fig.2 also shows the displacements for the case of compression of a strip by parallel slabs (curve *4*). We emphasize that the shape of the residual impression differs significantly from the shape of the stamp. We draw attention to the fact that the maximum depth of the impression is practically identical in all the cases although the shape of the impressions differ substantially. This result enables us to conclude that for sufficiently shallow stamps the maximum depth of the residual impression is independent of the shape of the stamp.

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# ON THE STABILITY OF THE LINING OF A HORIZONTAL OPENING IN A VISCOELASTIC AGEING MEDIUM*

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The stability of a long elastic tube in a viscoelastic medium is studied. Stability conditions, formulated in terms of the characteristics of the tube and the medium, are set up. Such probelms are of interest in studying the stability of underground structures /1-3/. The stability problem for a tube in the case when the medium is elastic was studied in /4/. This paper touches on the investigations in /5,6/.

**1. Formulation of the problem.** At a depth $H$ from the daylight surface in mountain rock, let there be a working (opening) of circular cross-section of radius $R$. The rock is considered to be a homogeneous, isotropic, viscoelastic medium filling the half-space. The working is reinforced, i.e., an elastic cylinder is imbedded which is fixed to the material of the rock surrounding the working. The lining is considered to be a homogeneous elastic medium. Far from the ends of the working, plane strain is realized in the rock and the lining. According to /7/, for $H/R > 50$ the problem of determining the state of stress and strain of the lining can be simplified and the lining can be considered as an elastic tube reinforcing a cylindrical hole in a viscoelastic space which is compressed by the uniform forces $p_1 = \gamma H$, $p_2 = \nu(1-\nu)^{-1}\gamma H$ far from the hole, where $\gamma$ is the specific gravity, and $\nu$ is Poisson's ratio of the rock.

Let the viscoelastic medium occupy all three-dimensional space. Let $x_1, x_2, x_3$ denote the coordinates of points of the medium in a Cartesian coordinate system $Ox_1x_2x_3$. A cylinder $x_1^2 + x_2^2 \leqslant 1$ is cut out of the medium, where the radius can be taken to be equal to unity without loss of generality. A circular elastic tube whose external radius equals unity is inserted into the hole being obtained. At the time $t = 0$ compressive forces of constant intensity $p_1$ along the $Ox_1$ axis and $p_2$ along the $Ox_2$ axis are applied to the viscoelastic medium at infinity, and a force of intensity $g$ directed perpendicular to the tube axis is applied to the inner surface of the tube. We introduce the cylindrical coordinate system $Or\vartheta x_3$, whose axis $Ox_3$ coincides with the tube axis, while the polar angle $\vartheta$ is measured from the $Ox_1$ axis. The forces applied to the inner surface of

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